NUCLEAR SCIENCE AND ENGINEERING: 26, 378-384 (1966)

Stability Analysis of a Sampled-Data Controlled Nuclear Reactor System

Frigyes Reisch

AB Atomenergi, Stockholm, Sweden

Received September 27, 1965

In a digital computer controlled system it is possible to monitor several variables almost at the same time and control the system according to the most critical one. This is called a sampled-data control system.

The purpose of the paper is to demonstrate how to handle such a problem. A simplified reactor system including neutron kinetics and fuel and cooling medium kinetics with a simple control circuit is examined. It is assumed that the reactor has a great number of cooling channels, and it is necessary to check the exit temperatures of the cooling medium as the maximum value is the limiting factor. Sampling is performed to accomplish this. The temperatures are scanned and a comparison is made between the value stored in the memory and the point being measured. The higher of the two values remains in the one word memory. After checking all the temperatures, a pulse representing the temperature of the hottest channel is sent through the sampler to the regulator and the memory is cleared.

A suitable method to study the stability is the z transform analysis. The procedures and logic followed are outlined here. First, the system is defined in the terms of Laplace transformation. Then the solving of the sampled system problem by the z transform theory is shown. A digital computer program is developed. The results of several calculations show the importance of choosing the right parameter combinations.

THE DESCRIPTION OF THE SYSTEM

The Sampled-Data Control

In this work it is assumed that the reactor has a great number of cooling channels, and it is necessary to check the exit temperatures of the cooling medium as the maximum value is the limiting factor.

Sampling is performed to accomplish this[1]. The temperatures are scanned and a comparison is made between the value stored in the memory and the point being measured. The higher of the two values remains in the one word memory which is a zero-order hold device. After the checking cycle, a pulse representing the exit temperature in the hottest channel is sent through the sampler to the regulator. Then the memory is cleared and the scanning procedure starts again. Thus, the scanning frequency is equal to the sampling frequency multiplied by the number of the cooling channels. This memory method implies a variable time delay or process lag. As many points are sampled, one never knows when the highest temperature is reached. To not lose any information during the transients, the sampling frequency must be more than twice as high as the highest frequency of the continuous system.

The scanning equipment, the sampler, and the regulator can be built together as a digital computer, which provides the input signal to the control rod mechanism. A perturbation reactivity signal is supplied to the reactor control system shown in Fig. 1 to illustrate how this feedback controlled system will perform. For the sake of simplicity, the control rods and regulator (K) are assumed to work as an integrator. With Laplace notations, one can write

$$K(s) = k_1/s \quad . \tag{1}$$

[1]J. T. TOU, *Digital and Sampled-data Control Systems*, McGraw-Hill Book Company Inc., New York, Toronto, London (1959).

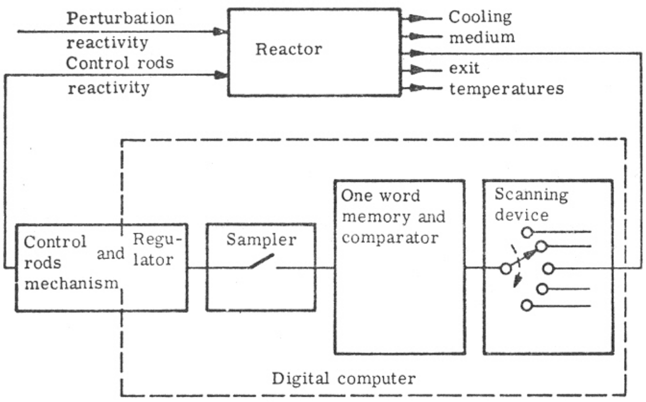


Fig. 1. The feedback controlled reactor system.

Here k_1 is the regulator constant. As the control block is an integrator receiving pulses it works as a zero-order hold device with stepwise outputs until the deviation of the output signal from a predetermined reference value is zero.

The Reactor

The purpose of this paper is to demonstrate the procedures and logic used when studying the stability problem of a sampled-data controlled reactor system with the z transform theory. It is not intended to present a particular solution; therefore, the internal dynamics of the reactor are simplified.

Reactivity perturbation (ρ_p) and control rod reactivity (ρ_k) are the input signals, and the exit temperatures of the cooling medium in the different channels (θ_c) are the output signals.

The reactor representation includes neutron kinetics (N), fuel kinetics (F), an internal feedback caused by the Doppler effect in the fuel (α), and the kinetics of the cooling medium (D). Figure 2 shows the relationship between the blocks creating the reactor.

The transfer function of the reactor with the usual Laplace transform notations (s) has the following form[2]

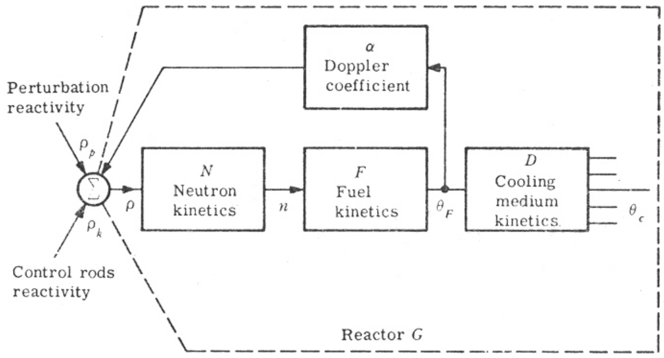
$$G(s) = \frac{\theta_c(s)}{\rho_p(s)} = \frac{N(s)\,F(s)\,D(s)}{1 + \alpha\,N(s)\,F(s)} \quad . \tag{2}$$

Neutron Kinetics

After linearizing the usual neutron kinetics equations, the following formula is obtained if only one delayed-neutron group is considered

$$N(s) = \frac{n(s)/n_0}{\rho\,(s)} = \frac{n_1}{s}\,\frac{1 + n_2 s}{1 + n_3 s} \quad . \tag{3}$$

[2]J. M. HARRER, *Nuclear Reactor Control Engineering*, D. Van Nostrand Company, Inc., Princeton, New Jersey, Toronto, London, New York (1963).



ρ = reactivity

n = neutron flux

θ_F = fuel temperature

θ_c = cooling medium temperature

Fig. 2. Reactor dynamics components.

Here,

$$n_1 = 1/(\ell + \beta/\lambda), \quad n_2 = 1/\lambda, \quad n_3 = 1/(\lambda + \beta/\ell), \tag{4}$$

where $1/\lambda$ is the mean lifetime of the delayed-neutron group and β is the fraction of delayed neutrons, ℓ is the lifetime of the neutrons; $n(s)/n_0$ and $\rho(s)$ are the relative changes of the nuclear power and the reactivity during transients, respectively.

Fuel and Cooling Medium Kinetics

It will be assumed that both the fuel and the cooling medium kinetics can be described by a simple time lag

$$F(s) = \frac{\theta_F(s)}{n(s)/n_0} = \frac{f_1}{1 + f_2 s} \tag{5}$$

$$D(s) = \frac{\theta_c(s)}{\theta_F(s)} = \frac{d_1}{1 + d_2 s} \quad , \tag{6}$$

where $\theta_F(s)$, f_2, f_1, and $\theta_c(s)$, d_2, d_1 are the transient temperature variation, the time constant, and the proportionality factor of the fuel and the cooling medium, respectively.

The Complete Plant

The complete plant is represented in Fig. 3. To make the mathematical reasoning easier, the block diagram can be condensed to include only the reactor block (G), the control block (K), and the sampler with sampling time T. See Fig. 4.

The functioning of the continuous system can be described by considering a temperature from one cooling channel.

The transfer function of the continuous plant (P) is the following

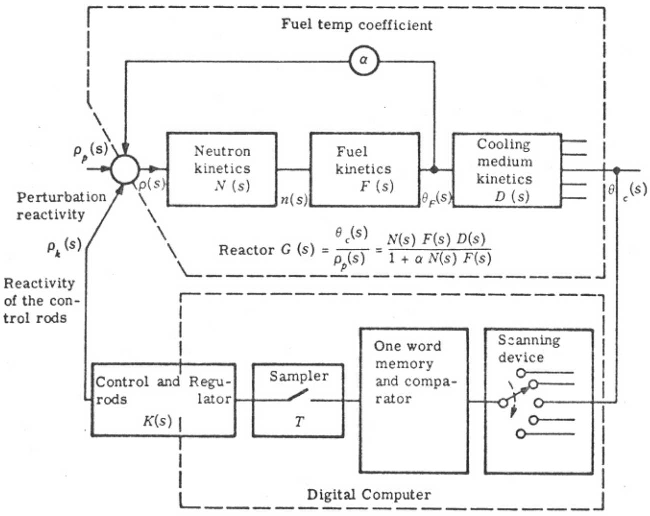


Fig. 3. Block diagram of the digital computer controlled simplified reactor system.

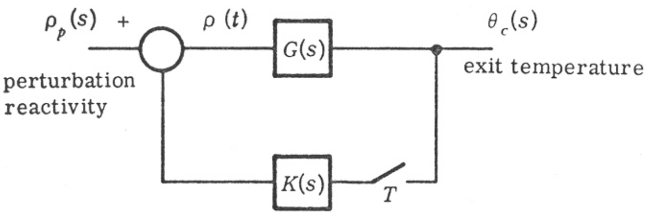


Fig. 4. The condensed block diagram of the sampled-data controlled system.

$$P(s) = \frac{\theta_c(s)}{\rho_p(s)} = \frac{G(s)}{1 + G(s)\,K(s)} \quad . \tag{7}$$

The denominator of $P(s)$ can be recognized as a typical characteristic equation. For a stable plant, the real part of all the roots of the characteristic equation must be negative, i.e., situated left from the imaginary axes in the complex s plane.

The weighting function or impulse response $w(t)$ of the continuous system in the time domain can be obtained by the inverse Laplace transformation

$$w(t) = L^{-1}\,[P(s)] \quad . \tag{8}$$

The frequency response can be obtained by the substitution

$$s \rightarrow j\omega \; . \tag{9}$$

Thus, the frequency response of the continuous system is

$$P(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)\,K(j\omega)} \quad . \tag{10}$$

The next step in the analysis should be the checking of the stability of the sampled-data controlled system. By z transformation, one can observe the system stability. One can also calculate, for example, a step response in the time domain by inverse z transformation. Before considering the application of the z transformation on this particular problem, some of the basic principles of the z transform theory will be reviewed.

SOME BASIC PROPERTIES OF THE z TRANSFORM THEORY

Consider a sampler as represented by Fig. 5. The input signal $x(t)$ is continuous while the output is pulsed. The pulsed or sampled value of $x(t)$ is $x^*(t)$; $x^*(t)$ is a noncontinuous function appearing as a pulse at time $t = T, 2T, \ldots$. Mathematically, $x^*(t)$ can be described with Dirac's delta function $\delta(t)$

$$x^*(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad . \tag{11}$$

The Laplace transform of the pulsed signal is $x^*(s)$. Using the usual Laplace transform theory, $x^*(s)$ can be calculated

$$x^*(s) = L\,[x^*(t)] = \sum_{n=0}^{\infty} x(nT)\exp(-nTs) \quad . \tag{12}$$

Equation (12) indicates that $x^*(s)$ is an infinite series in e^{Ts}; therefore, use is made of the substitution

$$s = \frac{\ln z}{T} \quad . \tag{13}$$

The z transform of $x^*(t)$ is thus

$$x(z) = z\,[x^*(t)] = L[x^*(t)]_s \quad , \tag{14}$$

where $s = (1/T)\ln z$.

There are extensive tables for z transforms containing functions commonly occurring in dynamic analysis.

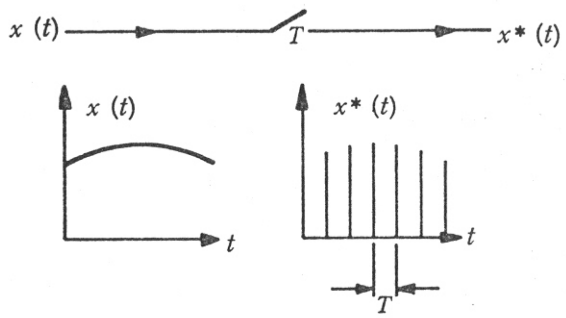


Fig. 5. The input and output signals of a sampler with sampling time T.

The use of the z transform is somewhat similar to the Laplace transform. See Fig. 6. Suppose that the system $P(s)$ has the sample-data input signal $x^*(s)$ and its continuous output signal is $y(s)$. If we sample $y(s)$ we get $y^*(s)$, i.e., the y values at the sampling times. These y values can be represented by the z transform theory

$$y(z) = x(z)\, P(z) \quad . \tag{15}$$

The z transform theory is valid only when the samplings frequency

$$f = 1/T \tag{16}$$

is at least twice as high as the highest frequency occurring in the frequency characteristic of the continuous system $P(j\omega)$.

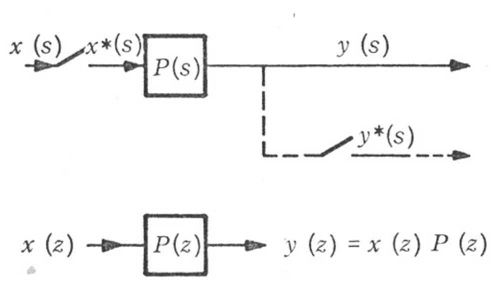


Fig. 6. A basic sampled-data system represented by z transform technique.

Some Stability Considerations

The z transformation is a nonlinear procedure. This means that to judge the stability of a system it is not enough to study only the plant $P(z)$ but the perturbation ρ_p must be taken in account too. The z transform of the output signal $\theta_c(z)$ depends on where the sampler is placed in the feedback loop. In this case, it is placed in the feedback path before the control block as it is shown in Fig. 4. The equation that should be used is

$$\theta_c(z) = \frac{G\rho_p(z)}{1 + GK(z)} \quad , \tag{17}$$

Where $G\rho_p(z)$ and $GK(z)$ are the z transforms of the respective Laplace functions

$$G\rho_p(z) = z\,[G(s)\,\rho_p(s)], \quad GK(z) = z\,[G(s)\,K(s)] \quad . \tag{18}$$

The characteristic equation in the z domain is the denominator of $\theta_c(z)$

$$1 + GK(z) = 0. \tag{19}$$

For a stable sampled-data controlled system, the roots of the characteristic equation must be inside the unit circle in the complex z plane.

For a stable system, the output is oscillatory with diminishing amplitudes if there are conjugate complex roots or real roots in the left half of the unit circle. If all the roots are on the positive real axis in the unit circle the response decays without oscillations.

The Transient Response in the Time Domain

Inverse z transformation may be carried out from the real inversion integral by partial-fraction expansion or by power-series expansion.

The third method, sometimes called long division, will be used in this particular case. Suppose that the z transform of the output signal can be written in the following manner

$$\theta_c(z) = \frac{z^0 v_0 + z^1 v_1 + z^2 v_2 + \ldots + z^n v_n}{z^0 w + z^1 w_1 + z^2 w_2 + \ldots + z^m w_m}$$

$$= z^0 c_0 + z^{-1} c_1 + \ldots$$

$$n \leq m \quad . \tag{20}$$

The θ_c values in the time domain at the sampling instances are $c_0, c_1, \ldots$. This means that even if the $\theta_c(t)$ function is continuous one can calculate the output signal only at $t = T, 2T, \ldots$.

To determine the complete transient response of a sampled-data system, modified z transform and modified inverse z transform technique have to be used, but these theories will not be discussed in this paper.

STABILITY ANALYSIS OF THE SAMPLED DATA CONTROLLED REACTOR SYSTEM WITH THE z TRANSFORM TECHNIQUE

All the principal equations necessary for the stability analysis of this system have been given. Following is a description of the particular problem.

The first aim is to find the z transform of the output signal $\theta_c(z)$ by using Eq. (17). Put the actual equations of the neutron (N), fuel (F), and cooling medium (D) kinetics in the reactor equation (G), i.e., combine Eqs. (2), (3), (5), and (6). The result[3] will be a fraction with a first-degree numerator and a fourth-degree denominator in "s". To make further calculations easier, suppose that the denominator has only real roots. They are $-a_1$, $-a_2$, $-a_3$, and $-a_4$. To determine the numerator of Eq. (17), the shape of the input signal has to be selected. Consider a step reactivity perturbation with a final value ρ_{p1}

$$\rho_p(s) = \rho_{p1}/s \quad . \tag{21}$$

[3]F. REISCH, "Stability Analysis of a Sampled-data Controlled Reactor System," AE-RTR-151, AB Atomenergi, Stockholm, Sweden (1965).

The numerator of Eq. (17) can now be written

$$G\rho_p(z) = z\,[G(s)\,\rho_p(s)] = \rho_{p1}\,\frac{n_1 n_2 f_1 d_1}{n_3 f_2 d_2 a_1 a_2 a_3 a_4}$$

$$\times\, z\left[\frac{a_1 a_2 a_3 a_4\ (s + 1/n_2)}{s(s + a_1)(s + a_2)(s + a_3)(s + a_4)}\right] . \quad (22)$$

Replace ρ_{p1} with k_1, and the function $GK(z)$ appearing in the denominator of Eq. (17) will be similar to Eq. (22).

To proceed, the z transformation indicated in Eq. (22) has to be performed. The easiest way is to use a z transform table where it can be found. This expression may be assumed to be equal to the fraction $H(z)/B(z)$

$$z\left[\frac{a_1 a_2 a_3 a_4\ (s + 1/n_2)}{s(s + a_1)(s + a_2)(s + a_3)(s + a_4)}\right] = \frac{H(z)}{B(z)} \quad (23)$$

$$= \frac{z/n_2}{z - 1} - x_1\,\frac{z}{z - e_1} - x_2\,\frac{z}{z - e_2} - x_3\,\frac{z}{z - e_3} - x_4\,\frac{z}{z - e_4} . \quad (24)$$

The effect of the sampling time can be seen if one realizes that

$$e_1 = \exp(-a_1 T),\ e_2 = \exp(-a_2 T),$$

$$e_3 = \exp(-a_3 T),\ \text{and}\ e_4 = \exp(-a_4 T) . \quad (25)$$

The remaining constants are

$$x_1 = \frac{a_2 a_3 a_4\ (-a_1 + 1/n_2)}{(a_2 - a_1)(a_3 - a_1)(a_4 - a_1)}$$

$$x_2 = \ldots\ x_3 = \ldots\ x_4 = \ldots \quad . \quad (26)$$

By rearranging Eq. (24), $H(z)$ and $B(z)$ will have the form of a power series

$$\frac{H(z)}{B(z)} = \frac{z^5 h_5 + z^4 h_4 + z^3 h_3 + z^2 h_2 + z^1 h_1 + z^0 h_0}{z^5 b_5 + z^4 b_4 + z^3 b_3 + z^2 b_2 + z^1 b_1 + z^0 b_0} . \quad (27)$$

Here, the constants $h_5 \ldots h_0$ and $b_5 \ldots b_0$ are the functions of n_2, $x_1 \ldots x_4$ and $a_1 \ldots a_4$, too lengthy to be written here.

The combination of Eqs. (17), (23), and (27) leads to a closed expression for the output signal in the form of a power series in z, which was the original goal

$$\theta_c = \frac{z^5 v_5 + z^4 v_4 + z^3 v_3 + z^2 v_2 + z^1 v_1 + z^0 v_0}{z^5 w_5 + z^4 w_4 + z^3 w_3 + z^2 w_2 + z^1 w_1 + z^0 w_0} . \quad (28)$$

Here, the constants $v_5 \ldots v_0$ and $w_5 \ldots w_0$ are the functions of all the parameters. They are not given here because of their length, but can be derived from the previous formulas.

As was previously stated, the roots of the denominator of Eq. (28) must be inside the unit circle for a stable system. The step response in the time domain can be calculated by long division.

Calculation with the Digital Computer Program

It is quite clear that to perform all the calculation by hand would be hopelessly time consuming. Because of this, a digital computer program has been developed. The program gives the frequency functions of the reactor $G(j\omega)$ and the plant $P(j\omega)$. The roots of the characteristic equation of $\theta_c(z)$ and the step response in the time domain are calculated, too.

Numerical Examples[a]

Suppose the following parameters for the case a):

neutron kinetics	ℓ = 1 msec, β = 0.65%, $1/\lambda$ = 77 msec
fuel	f_1 = 15°C/%, f_2 = 5 sec, α = 3 pcm/deg C
cooling medium	d_1 = 0.3°C/deg C, d_2 = 6 sec (including the sensing device)
regulator constant	k_1 = 1.0 pcm/sec deg C
samplings frequency	1/T = 10 counts/sec
perturbation reactivity	+ 50 pcm step

Case b) will have the same parameters except the regulator constant k_1 = 0.01 pcm/sec deg C.

The continuous systems' amplitude characteristics from the frequency responses are drawn in Fig. 7. As the sampling frequency is 10 counts/sec, all the amplitudes belonging to the frequencies higher than half of the samplings frequency 5 counts/sec were neglected. The justification is that all these amplitudes are less than 1% of the largest amplitude $|P(j\omega)|_{max}$.

The next step is to study the characteristic equation in the z plane. For case a) the roots are

No. 1	0.9923
No. 2	0.7914
No. 3	0.6451
No. 4	0.9964 + 0.0467 j
No. 5	0.9964 - 0.0467 j

[a]In these examples, reactivity is expressed in percent mil (pcm) where 1 pcm $= 10^{-5}\ \Delta k/k$. In this case, 1 dollar = 650 pcm.

and for case b)

No. 1	0.9989
No. 2	0.7973
No. 3	0.6478
No. 4	0.9845
No. 5	0.9932 .

The roots for both cases are inside the unit circle; therefore, the system is stable. See Fig. 8.

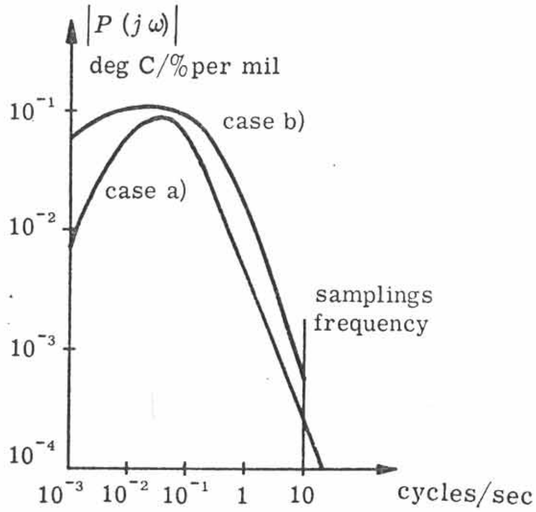


Fig. 7. The frequency response of the system without sampling.

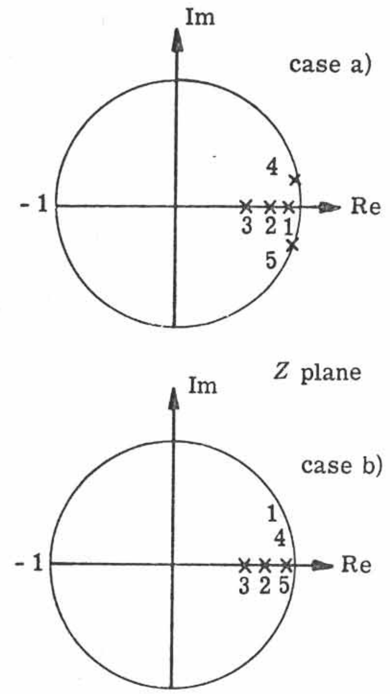


Fig. 8. The roots of the characteristic equations in the z domain for systems with

case a) oscillatory diminishing response
case b) exponentially decaying response.

In case a), the step response is oscillatory as there are two complex conjugated roots. In case b), all the roots are on the positive real axis indicating an exponentially decaying step response in the time domain.

The step responses in the time domain calculated by long division justify the predictions as shown in Fig. 9.

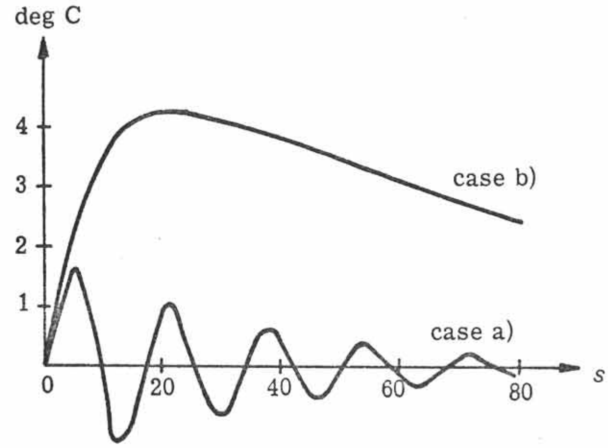


Fig. 9. The time domain step response for systems with oscillatory and exponential decay.

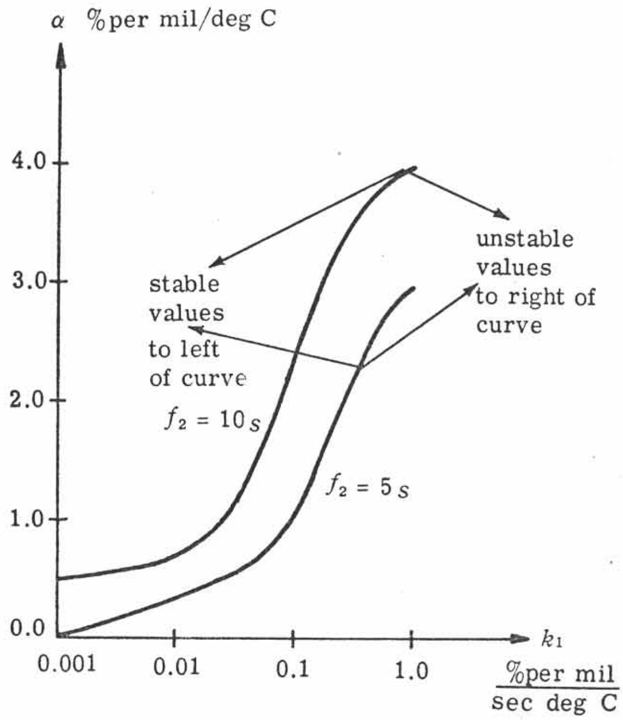


Fig. 10. Stability region for varying fuel temperature coefficient (α) and regulator constant (k_1). With two different fuel time constants (f_2 = 5 sec, f_2 = 10 sec). Reactivity perturbation = +10 pcm step.

If some of the parameters are uncertain, it is necessary to chart a stability diagram. Take the previous case and suppose that the fuel temperature coefficient and time constant are known with insufficient accuracy

$$1 \text{ pcm/deg C} \lesssim \alpha \lesssim 3 \text{ pcm/deg C}$$

$$5 \text{ sec} \lesssim f_2 \lesssim 10 \text{ sec} .$$

The task is to choose a control constant k_1. The expected step reactivity perturbation is + 10 pcm.

As a first step, it is sufficient to calculate the roots of the characteristic equation with different α and k_1 values for f_2 = 5 and 10 sec. From these data one can draw a curve similar to those in Fig. 10. The (α, k_1) plane is divided by the curve f_2= constant. To the left of the curve, there are the α, k_1 values which give a stable system, e.g., f_2= 10 sec, α = 1.5 pcm/deg C and k_1 = 0.01 pcm/sec deg C. To the right of the curve, there are the combinations which give unstable responses, e.g., f_2 = 10 sec, α = 1.0 pcm/deg C and k_1 = 0.1 pcm/sec deg C.

It should be understood that, in the stable region, the nearer the parameters are to the stability limit the more oscillatory the system becomes, but at the same time the temperature error is more quickly corrected. As might be expected, a faster fuel time constant (f_2= 5 sec) has a larger stability region. Before choosing a final value for k_1 and determining the permissible maximum and minimum values, it is necessary, of course, to study the overshoots of the temperature transients in the time domain. This can be performed according to the procedure indicated in the numerical examples previously given.

ACKNOWLEDGMENTS

As this work was not the author's main job, it was completed during a number of short periods. In the different working phases, A. Ponteus, S. Palsson, and B. Furugren were helpful.